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We introduce the theory of operator monotone functions and employ it to derive a new inequality relating the quantum relative entropy and the quantum conditional entropy. We present applications of this new inequality and in particular we prove a new lower bound on the relative entropy of entanglement and other properties of entanglement measures.

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Recently the entanglement of finite systems has received considerable attention [1,2] when it was realized that the theory of majorization [3,4] provides a simple mathematical framework in which the theory can be formulated [1]. In general the well developed theory of matrix analysis provides many techniques and ideas that may be useful for the study of entanglement. However, the restriction to finite entanglement, while justified from an experimental point of view, places an additional constraint on the system which may cloud some of the truly fundamental aspects of entanglement. Therefore the study of the asymptotic limit, i.e. a situation in which large numbers of entangled pairs can be manipulated simultaneously, is of interest from a fundamental point of view. A substantial body of work has been developed in recent years, beginning with the case of pure entangled states [5] and an extensive study of different ways to quantify the amount of entanglement in mixed bipartite states. Some interesting examples are the entanglement of formation [6,7], the entanglement of distillation, and the relative entropy of entanglement [8–11]. With the notable exception of the entanglement of formation [7], these entanglement measures are very difficult to compute analytically even in the qubit case. Therefore it is of great interest to obtain upper and lower bounds for them. To further our understanding of entanglement and our ability to manipulate it locally, it is of interest to try to establish connections to other ideas such as distinguishability [12,9,10] or thermodynamical considerations [13,14]. In these contexts one mathematical function emerges as a central quantity, namely the relative entropy which is defined as

$$S(\sigma||\rho) = \text{tr}\{\sigma \log \sigma - \sigma \log \rho\} . \quad (1)$$

It has a number of remarkable properties [8,12,15] and is closely related to the problem of the quantification of entanglement [8–11], the distinguishability of quantum states [9,10] and to thermodynamical ideas (see for example [16]). Any new inequality relating the relative entropy to other entropic quantities is therefore expected to lead to potentially important new insights into any of these topics and is potentially an important contribution.

In general one would attempt to formulate inequalities

that are valid for any density operator. For the study of entanglement, however, a particular special type of inequality would be of great interest. These are inequalities that are only valid when at least some of the density operators that are involved are non-distillable or separable, but may be violated for distillable states. These inequalities naturally lead to sufficient criteria for the distillability of states and they are, as we will demonstrate here, very useful for example in the study of entanglement measures.

In this paper we combine the ideas of positive maps [20,21] with the concept of operator monotones which has been developed originally in matrix analysis [4] to derive such a new inequality relating the relative entropy and the entropy. We present some lemmas and corollaries to this inequality to demonstrate its usefulness. In particular we derive a new lower bound on the relative entropy of entanglement and a much simplified proof that for pure states the relative entropy of entanglement coincides with the entropy of entanglement.

Let us briefly introduce the idea of operator monotone function as this is a concept which is not very familiar to quantum information theory. Much more material can be found for example in [4]. First we begin with

Definition 1 *Given two operators A and B , we say that $A \geq B$ if the operator $A - B$ is a non-negative operator, i.e. $A \geq B$ if for all $|\psi\rangle$ we have $\langle\psi|A - B|\psi\rangle \geq 0$.*

This definition allows us to compare operator and in particular we are now able to introduce the idea of operator monotones. Given a real valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ we canonically extend it to a function on Hermitean operators [4]. Then we make the following

Definition 2 *A function f is called **operator monotone**, if for all pairs of Hermitean operators satisfying $A \geq B$ we have $f(A) \geq f(B)$.*

It should be noted that ordinary monotonicity of a function and operator monotonicity are two entirely different concepts. An example is the function $f(x) = x^2$ on the interval $[0, \infty]$, which is not an operator monotone function although it is clearly a monotone function in the

ordinary sense [4]! In physics, and in particular in thermodynamics and the theory of entanglement the entropy and therefore the logarithm plays a central role. It is therefore important to note that

Lemma 3 *The function $f(x) = \log(x)$ is operator monotone!*

The complicated proof of this theorem can be found in [4,17]. Lemma 3 is one of the key ingredients in the proof of our new inequality.

The other major input comes from the theory of positive but not completely positive maps, whose application to quantum entanglement of mixed states was pioneered by the Horodeckis and further developed for example in [21]. Positive maps can be used to detect the non-separability of mixed states and a number of important positive maps have been found, amongst them the well known partial transposition [18,19]. Here we employ a different positive map which has been introduced in [20]. This map, the reduction map Λ , is defined as

$$\Lambda(\rho) := \mathbf{1} \text{tr} \rho - \rho . \quad (2)$$

The reduction map is evidently positive, but not completely positive as the map $\mathbf{1} \otimes \Lambda$ is not positive, i.e. it can transform a positive operator into a non-positive operator. The corresponding reduction criterion is then given by

$$\rho \text{ is non-distillable} \Rightarrow \rho_A \otimes \mathbf{1} \geq \rho_{AB} . \quad (3)$$

The reduction criterion is remarkable as its violation implies distillability of the density operator ρ_{AB} while this is not known to be the case for the partial transposition.

Now we use the two key properties of operator monotonicity of the logarithm (Lemma 3) and the reduction criterion Eq. (3) to prove

Theorem 4 *For any non-distillable state ρ_{AB} and for any state σ_{AB} of a bipartite system AB we have*

$$S(\sigma_A) - S(\sigma_{AB}) \leq S(\sigma_{AB}||\rho_{AB}) - S(\sigma_A||\rho_A) , \quad (4)$$

$$S(\sigma_B) - S(\sigma_{AB}) \leq S(\sigma_{AB}||\rho_{AB}) - S(\sigma_B||\rho_B) . \quad (5)$$

It should be noted that the left hand side of the inequality is the negative conditional entropy which is negative for all separable states σ_{AB} , while it can take positive values for entangled states (an example is the singlet state).

Before we discuss the implications of this theorem further let us present its proof.

Proof: Given a non-distillable state ρ_{AB} , the reduction criterion and the operator monotonicity of the logarithm imply that [22]

$$\log(\rho_A \otimes \mathbf{1}_B) \geq \log(\rho_{AB}) .$$

This statement is equivalent to

$$\begin{aligned} & \forall \sigma_{AB} : \text{tr}\{\sigma_{AB} \log(\rho_A) \otimes \mathbf{1}_B\} \geq \text{tr}\{\sigma_{AB} \log \rho_{AB}\} \\ & \Leftrightarrow \forall \sigma_{AB} : -\text{tr}\{\sigma_{AB} \log(\rho_A) \otimes \mathbf{1}_B\} \leq -\text{tr}\{\sigma_{AB} \log \rho_{AB}\} \\ & \Leftrightarrow \forall \sigma_{AB} : -\text{tr}\{\sigma_A \log \rho_A\} \leq -\text{tr}\{\sigma_{AB} \log \rho_{AB}\} . \end{aligned}$$

To draw the connection to the relative entropy we use Definition 1 to find the equivalent statement

$$\begin{aligned} & \forall \sigma_{AB} : -S(\sigma_{AB}) + S(\sigma_A) - S(\sigma_A) - \text{tr}\{\sigma_A \log \rho_A\} \leq \\ & \quad -S(\sigma_{AB}) - \text{tr}\{\sigma_{AB} \log \rho_{AB}\} \\ & \Leftrightarrow \forall \sigma_{AB} : -S(\sigma_{AB}) + S(\sigma_A) + S(\sigma_A||\rho_A) \leq S(\sigma_{AB}||\rho_{AB}) \\ & \Leftrightarrow \forall \sigma_{AB} : S(\sigma_A) - S(\sigma_{AB}) \leq S(\sigma_{AB}||\rho_{AB}) - S(\sigma_A||\rho_A) . \end{aligned}$$

Interchanging the roles of A and B we find the second inequality \square

In the following we present some applications of the new inequality presented in Theorem 4. Firstly, let us demonstrate that from Theorem 4 we can obtain a new lower bound on the relative entropy of entanglement. We find

Lemma 5 *The relative entropy of entanglement is bounded from below by the negative conditional entropy, i.e. for all σ_{AB} we have*

$$E_{RE}(\sigma_{AB}) \geq \max\{S(\sigma_A) - S(\sigma_{AB}), S(\sigma_B) - S(\sigma_{AB})\} .$$

Proof: The relative entropy of entanglement is defined as

$$E_{RE}(\sigma_{AB}) = \min_{\rho_{AB} \in \mathcal{D}} S(\sigma_{AB}||\rho_{AB}) \quad (6)$$

where \mathcal{D} either denotes the set of separable states [8,10], the set of states with positive partial transpose [11] or the set of non-distillable states [23]. Lemma 5 applies to both definitions and we only prove the strongest one, choosing \mathcal{D} to be the set of non-distillable states. Let us denote by ρ_{AB}^* the non-distillable state that realizes the relative entropy of entanglement, i.e.

$$E_{RE}(\sigma_{AB}) = S(\sigma_{AB}||\rho_{AB}^*) . \quad (7)$$

From Theorem 4 and the non-negativity of the relative entropy we conclude that

$$\begin{aligned} S(\sigma_A) - S(\sigma_{AB}) & \leq S(\sigma_{AB}||\rho_{AB}^*) - S(\sigma_A||\rho_A^*) \\ & \leq S(\sigma_{AB}||\rho_{AB}^*) \\ & = E(\sigma_{AB}) . \end{aligned} \quad (8)$$

Interchanging systems A and B and combining the result yields the Lemma 5 \square

A direct consequence of Theorem 1 is a relationship between the relative entropy of entanglement and the entanglement of formation.

Corollary 6 For any bipartite state σ_{AB} we have

$$E_{RE}(\sigma_{AB}) \geq E_F(\sigma_{AB}) - S(\sigma_{AB}) .$$

Proof: This follows immediately from Lemma 5 because $E_F(\sigma_{AB}) \leq S(\sigma_A)$ \square

A remarkable consequence of this Theorem 1 is a very simple proof, that the relative entropy of entanglement for pure states coincides with the entropy of entanglement, i.e. the entropy of the reduced density operator of one subsystem. This statement was first proven in [10], however, these proofs are very complicated. Using Lemma 5 this proof is simplified considerably.

Corollary 7 For pure states $|\psi_{AB}\rangle$ we find

$$E_{RE}(|\psi_{AB}\rangle\langle\psi_{AB}|) = S(\rho_A) . \quad (9)$$

where $\rho_A = \text{tr}_B\{|\psi_{AB}\rangle\langle\psi_{AB}|\}$.

Proof: Up to local unitary operations we can write $|\psi_{AB}\rangle = \sum_{i=1}^n \alpha_i |i\rangle_A |i\rangle_B$ for an orthonormal basis $\{|i\rangle\}_{i=1,n}$. For the mixed state $\rho_{AB} = \sum_{i=1}^n |\alpha_i|^2 |ii\rangle\langle ii|$ we find $E_{RE}(|\psi_{AB}\rangle\langle\psi_{AB}|) \leq S(|\psi_{AB}\rangle\langle\psi_{AB}| || \rho_{AB}) = S(\rho_A)$. On the other hand Lemma 1 provides $E_{RE}(|\psi_{AB}\rangle\langle\psi_{AB}|) \geq S(\rho_A)$ and therefore we conclude that Corollary 7 is correct \square

It is interesting to note that, to our knowledge, all states for which we know the distillable entanglement under local operations and one-way communication, it actually coincides with the negative conditional entropy and one may conjecture that indeed the distillable entanglement under local operations and one way communication is equal to the negative conditional entropy. A similar conjecture has been made by Rains for maximally correlated states [11].

Another small conclusion we can draw from Theorem 1 is the following

Lemma 8 For states σ_{AB} such that $E_{RE}(\sigma) = \max\{S(\sigma_A) - S(\sigma_{AB}), S(\sigma_B) - S(\sigma_{AB})\}$ the closest state $\rho_{AB}^* \in \mathcal{D}$ must have the same reduced density operator as σ_{AB} .

Proof: Without restricting generality we can assume $E_{RE}(\sigma_{AB}) = S(\sigma_A) - S(\sigma_{AB})$. This implies $S(\sigma_{AB} || \rho_{AB}^*) = S(\sigma_A) - S(\sigma_{AB})$. But from Theorem 4 we have $S(\sigma_{AB} || \rho_{AB}^*) - S(\sigma_A || \rho_A^*) \geq S(\sigma_A) - S(\sigma_{AB})$ which implies $S(\sigma_A || \rho_A^*) = 0$ and therefore $\sigma_A = \rho_A^*$ \square

It is important to note, that the lower bound derived in Lemma 5 is actually additive. This allows to draw some conclusions concerning the additivity of the relative entropy of entanglement. In fact, for density operators that achieve the lower bound presented in Lemma 5 the relative entropy is additive.

Lemma 9 If two density operators ρ_1 and ρ_2 both satisfy $E_{RE}(\rho_i) = S(\rho_{i,A}) - S(\rho_{i,AB})$ then we have

$$E_{RE}(\rho_1 \otimes \rho_2) = E_{RE}(\rho_1) + E_{RE}(\rho_2) , \quad (10)$$

i.e. the relative entropy of entanglement is additive for ρ_1 and ρ_2 .

Proof: It is obvious that for any ρ_1 and ρ_2 we have $E_{RE}(\rho_1 \otimes \rho_2) \leq E_{RE}(\rho_1) + E_{RE}(\rho_2)$. But on the other hand $E_{RE}(\rho_1 \otimes \rho_2) \geq S(\rho_{1A} \otimes \rho_{2A}) - S(\rho_{1AB} \otimes \rho_{2AB}) = E_{RE}(\rho_1) + E_{RE}(\rho_2)$ because of additivity of the conditional entropy. Therefore $E_{RE}(\rho_1 \otimes \rho_2) = E_{RE}(\rho_1) + E_{RE}(\rho_2)$ \square

In summary we have proven a new inequality relating the quantum conditional entropy and quantum relative entropy. To demonstrate the usefulness of this inequality, we have used it to derive a new lower bound on the relative entropy of entanglement as well as some remarkably simple proofs of some other properties of the relative entropy of entanglement. Relations between different entanglement measures could be obtained from our inequality and we believe that this will lead to other useful applications in such diverse fields as entanglement, distinguishability or thermodynamics.

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